Holography, Unfolding and Higher-Spin Theories

arXiv:1203.5554

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Round Table

Frontiers of Mathematical Physics

Dubna, December 18 2012

HS gauge theory

Higher derivatives in interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots , \qquad S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

HS Gauge Theories $(m = 0)$: Fradkin, M.V. (1987)

$$AdS_d : \qquad [D_n, D_m] \sim \rho^{-2} = \lambda^2$$

Non-locality beyond any (=Plank) scale: Quantum Gravity?!

AdS/CFT:

$(3d, m = 0) \otimes (3d, m = 0) = \sum_{s=0}^{\infty}(4d, m = 0)$ Flato, Fronsdal (1978);

Sundborg (2001), Sezgin, Sundell (2002,2003), Klebanov, Polyakov (2002),

Giombi, Yin (2009). . . , Maldacena, Zhiboedov (2011,2012)

Results

CFT_3 **dual of** AdS_4 **HS theory: 3d conformal HS theory**

Holography: Unfolding

Plan

I Unfolded dynamics and holographic duality

II Free massless HS fields in AdS_4

III Conserved currents and massless equations

IV AdS_4 HS theory as $3d$ conformal HS theory

V Holographic duality of relativistic and non-relativistic theories

VI Conclusion

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \textbf{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional covariant generalization

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p}$$

$$\mathbf{dW^\Omega(x) = G^\Omega(W(x))\,, \qquad d = dx^n \partial_n}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Φ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \ldots \Phi_n} W^{\Phi_1} \wedge \ldots \wedge W^{\Phi_n}$$

$d > 1$: **Nontrivial compatibility conditions**

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

Any solution: FDA Sullivan (1968); D'Auria and Fre (1982)

The unfolded equation is invariant under the gauge transformation

$$\delta W^\Omega(x) = d\varepsilon^\Omega(x) + \varepsilon^\Phi(x) \frac{\partial G^\Omega(W(x))}{\partial W^\Phi(x)}\,,$$

Vacuum geometry

$\omega = \omega^{\alpha} T_{\alpha}$: h valued 1-form.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^{\alpha} \wedge \omega^{\beta}[T_{\alpha}, T_{\beta}]$$

the unfolded equation with $W = \omega$ has the zero-curvature form

$$d\omega + \omega \wedge \omega = 0\,.$$

Compatibility condition: Jacobi identity for h.

FDA: usual gauge transformation of the connection ω.

Zero-curvature equations: background geometry in a coordinate independent way.

If h is Poincare or anti-de Sitter algebra it describes Minkowski or AdS_d space-time

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms

  Exterior algebra formalism
- Interactions: nonlinear deformation of $G^{\Omega}(W)$
- Local degrees of freedom are in 0-forms $C^{i}(x_0)$ at any $x = x_0$ (as $q(t_0)$)

  infinite-dimensional module dual to the space of single-particle states
- Independence of ambient space-time

  Geometry is encoded by $G^{\Omega}(W)$

Unfolding and holographic duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time:

ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$d = dX^n \frac{\partial}{\partial X^n} \to \tilde{d} = dX^n \frac{\partial}{\partial X^n} + d\hat{X}^{\hat{n}} \frac{\partial}{\partial \hat{X}^{\hat{n}}}\,, \qquad dX^n W_n \to dX^n W_n + d\hat{X}^{\hat{n}} \hat{W}_{\hat{n}}\,,$$

$\hat{X}^{\hat{n}}$ are additional coordinates

$$\tilde{d} W^{\Omega}(X, \hat{X}) = G^{\Omega}(W(X, \hat{X}))$$

Two unfolded systems in different space-times are equivalent (dual) if they have the same unfolded form.

Direct way to establish holographic duality between two theories: unfold both to see whether their unfolded formulations coincide.

Particular space-time interpretation of a universal unfolded system, e.g, whether a system is on-shell or off-shell, depends not only on $G^{\Omega}(W)$ but, in the first place, on space-time M^d and chosen vacuum solution $W_0(X)$.

Given unfolded system generates a class of holographically dual theories in different dimensions.

Free massless fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2 \ldots$

Fermions require doubling of fields

$$\omega^{ii}(y,\bar{y}\mid x)\,,\qquad C^{i1-i}(y,\bar{y}\mid x)\,,\qquad i=0,1\,,$$

$$\bar{\omega}^{ii}(y,\bar{y}\mid x)=\omega^{ii}(\bar{y},y\mid x)\,,\qquad \bar{C}^{i\,1-i}(y,\bar{y}\mid x)=C^{1-i\,i}(\bar{y},y\mid x)$$

$$A(y,\bar{y}\mid x)=i\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\beta}_1}\ldots\bar{y}_{\dot{\beta}_m}A^{\alpha_1\ldots\alpha_n}{}_{,}{}^{\dot{\beta}_1\ldots\dot{\beta}_m}(x)$$

The unfolded system for free massless fields is **MV (1989)**

- ⋆ $$R_1^{ii}(y,\bar{y}\mid x)=\eta\,\overline{H}^{\dot{\alpha}\dot{\beta}}\frac{\partial^2}{\partial\bar{y}^{\dot{\alpha}}\partial\bar{y}^{\dot{\beta}}}\,C^{1-i\,i}(0,\bar{y}\mid x)+\bar{\eta}\,H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\,C^{i\,1-i}(y,0\mid x)$$
- ⋆ $$\tilde{D}_0 C^{i\,1-i}(y,\bar{y}\mid x)=0$$

$$R_1(y,\bar{y}\mid x)=D_0^{ad}\omega(y,\bar{y}\mid x)\qquad H^{\alpha\beta}=e^{\alpha}{}_{\dot{\alpha}}\wedge e^{\beta\dot{\alpha}}\,,\quad \overline{H}^{\dot{\alpha}\dot{\beta}}=e_{\alpha}{}^{\dot{\alpha}}\wedge e^{\alpha\dot{\beta}}$$

$$D_0^{ad}\omega=D^L-\lambda e^{\alpha\dot{\beta}}\left(y_{\alpha}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}}+\frac{\partial}{\partial y^{\alpha}}\bar{y}_{\dot{\beta}}\right)\,,\qquad \tilde{D}_0=D^L+\lambda e^{\alpha\dot{\beta}}\left(y_{\alpha}\bar{y}_{\dot{\beta}}+\frac{\partial^2}{\partial y^{\alpha}\partial\bar{y}^{\dot{\beta}}}\right)$$

$$D^L=d_x-\left(\omega^{\alpha\beta}y_{\alpha}\frac{\partial}{\partial y^{\beta}}+\bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}}\right)$$

Non-Abelian HS algebra

Star product

$$(f * g)(Y) = \int dS dT f(Y+S) g(Y+T) \exp -iS_A T^A$$

$$[Y_A, Y_B]_* = 2iC_{AB}\,, \qquad C_{\alpha\beta} = \epsilon_{\alpha\beta}\,, \qquad C_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}$$

Non-Abelian HS curvature

$$R_1(y,\bar y|x) \to R(y,\bar y|x) = d\omega(y,\bar y|x) + \omega(y,\bar y|x) * \omega(y,\bar y|x)$$

$$\tilde D_0 C(y,\bar y|x) \to \tilde D C(y,\bar y|x) = dC(y,\bar y|x) + \omega(y,\bar y|x) * C(y,\bar y|x) - C(y,\bar y|x) * \omega(y,-\bar y|x)$$

$3d$ conformal equations

Conformal invariant massless equations in $d=3$

$$dx^{\alpha\beta}\Big(\frac{\partial}{\partial x^{\alpha\beta}}\pm\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\Big)C(y|x)=0\,,\qquad \alpha,\beta=1,2$$

Shaynkman, MV (2001)

Rank r unfolded equations: tensoring of Fock modules Gelfond, MV (2003)

$$dx^{\alpha\beta}\Big(\frac{\partial}{\partial x^{\alpha\beta}}+\eta_{ij}\frac{\partial^2}{\partial y_i^\alpha\partial y_j^\beta}\Big)C(y|x)=0\,,\qquad i,j=1,\dots r\,.$$

For diagonal η^{ij} higher-rank equations are satisfied by

$$C(y_i|x)=C_1(y_1|x)C_2(y_2|x)\dots C_r(y_r|x)\,.$$

Rank-two equations: conserved currents

$$\left\{\frac{\partial}{\partial x^{\alpha\beta}}-\frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}}\right\}T(u,\,y|x)=0$$

$T(u,\,y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$T(u,\,y\,|x)=\sum_{i=1}^{N}C_{+\,i}(y-u|x)\,C_{-\,i}(u+y|x)$$

Rank-two fields: bilocal fields in the twistor space.

Dynamical currents (primaries)

$$J(u|x) = T(u,0\,|x)\,, \qquad \tilde{J}(y|x) = T(0,y|x)$$

Gelfond, MV (2003)

$$J^{asym}(u,y|x) = u_\alpha y^\alpha \Big(\frac{\partial^2}{\partial u^\beta \partial y_\beta} T(u,y|x)\Big|_{u=y=0} \Big)$$

$J(u|x)$ **generates** $3d$ **currents of all integer and half-integer spins**

$$J(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} J_{\alpha_1\dots\alpha_{2s}}(x)\,, \qquad \tilde{J}(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x)\,.$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1\dots\alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Differential equations: conventional conservation condition

$$\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u|x) = 0\,, \qquad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} \tilde{J}(y|x) = 0$$

$3d$ conformal setup in AdS_4 HS theory

For manifest conformal invariance introduce

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\,, \qquad y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)\,, \qquad [y^-_\alpha\,, y^{+\beta}]_* = \delta^\beta_\alpha$$

$3d$ conformal realization of the algebra $sp(4;\mathbb{R}) \sim o(3,2)$

$$L^\alpha{}_\beta = y^{+\alpha} y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma} y^-_\gamma\,, \qquad D = \frac{1}{2} y^{+\alpha} y^-_\alpha$$

$$P_{\alpha\beta} = i y^-_\alpha y^-_\beta\,, \qquad K^{\alpha\beta} = -i y^{+\alpha} y^{+\beta}$$

Conformal weight of HS gauge fields

$$[D, \omega(y^\pm|X)] = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} - y^-_\alpha \frac{\partial}{\partial y^-_\alpha}\right)\omega(y^\pm|X)\,.$$

Pullback $\widehat{\omega}(y^\pm|x)$ of $\omega(y^\pm|x)$ to Σ: $3d$ conformal HS gauge fields

Holography at infinity

AdS_4 **foliation:** $x^n = (\mathbf{x}^a, z)$: $\mathbf{x}^a$ **are coordinates of leafs** $(a = 0, 1, 2,)$ z **is a foliation parameter**

Poincaré coordinates

$$W = \frac{i}{z} d\mathbf{x}^{\alpha\beta} y^-_\alpha y^-_\beta - \frac{dz}{2z} y^-_\alpha y^{+\alpha}$$

$$e^{\alpha\dot\alpha} = \frac{1}{2z} dx^{\alpha\dot\alpha}\,, \qquad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta}\,, \qquad \bar\omega^{\dot\alpha\dot\beta} = \frac{i}{4z} d\mathbf{x}^{\dot\alpha\dot\beta}$$

$$\left[d_{\mathbf{x}} + \frac{i}{z} d\mathbf{x}^{\alpha\beta} \left(y_\alpha \frac{\partial}{\partial y^\beta} - \bar y_\alpha \frac{\partial}{\partial \bar y^\beta} + y_\alpha \bar y_\beta - \frac{\partial^2}{\partial y^\alpha \partial \bar y^\beta} \right) \right] C(y, \bar y | \mathbf{x}, z) = 0$$

Rescaling y^α and $\bar y^{\dot\alpha}$ **via**

$$C(y, \bar y | \mathbf{x}, z) = z \exp(y_\alpha \bar y^\alpha) T(w, \bar w | \mathbf{x}, z)\,,$$

$$w^\alpha = z^{1/2} y^\alpha\,, \qquad \bar w^\alpha = z^{1/2} \bar y^\alpha$$

$T(w, \bar w | \mathbf{x}, z)$ **satisfies the** $3d$ **conformal invariant current equation**

$$\left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^\alpha \partial \bar w^\beta} \right] T(w, \bar w | \mathbf{x}, z) = 0$$

Connections

Setting

$$W^{jj}(y^{\pm}|\mathbf{x},z) = \Omega^{jj}(v^-, w^+|\mathbf{x},z)$$

$$\mathbf{v^{\pm} = z^{-1/2} y^{\pm}}\,, \qquad \mathbf{w^{\pm} = z^{1/2} y^{\pm}}$$

manifest z–dependence disappears

$$D_{\mathbf{x}}\Omega^{jj}(v^-, w^+|\mathbf{x},z) = \left(d_{\mathbf{x}} + 2i d\mathbf{x}^{\alpha\beta} v^-_\alpha \frac{\partial}{\partial w^{+\beta}}\right)\Omega^{jj}(v^-, w^+|\mathbf{x},z)$$

Using

$$w_\alpha = w^+_\alpha + izv^-_\alpha\,, \qquad \bar{w}_\alpha = iw^+_\alpha + zv^-_\alpha$$

in the limit $z \to 0$ free HS equations take the form

$$\star \quad D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^-, w^+|\mathbf{x},0) = d\mathbf{x}_\alpha{}^\gamma d\mathbf{x}_{\beta\gamma}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}(w^+, 0 \mid \mathbf{x}, 0)\,,$$

$$\star \quad \left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{-\beta}}\right] T^{j\,1-j}(w^+, w^-|\mathbf{x},0) = 0\,.$$

$$\mathcal{T}^{jj}(w^+, w^-|\mathbf{x},0) = \eta T^{j\,1-j}(w^+, w^- \mid \mathbf{x},0) - \bar{\eta} T^{1-j\,j}(-iw^-, iw^+ \mid \mathbf{x},0)$$

Towards nonlinear $3d$ conformal HS theory

Conformal HS theory is nonlinear since conformal HS curvatures inherited from the AdS_4 HS theory are non-Abelian Fradkin, Linetsky (1990)

$$R_{\mathbf{x}\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) = d_{\mathbf{x}}\Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) + \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) \star \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x})$$

It is important

$$[v^-_\alpha, w^{+\beta}]_\star = \delta_\alpha^\beta$$

The equation on 0-forms deforms to nonlinear twisted adjoint representation

$$dT(w^\pm|x) + \Omega(\frac{\partial}{\partial w^{+\beta}}, w^+_\alpha) \circ T(w^\pm|x) - T(w^\pm|x) \circ \Omega(-i\eta\frac{\partial}{\partial w^{-\alpha}}, -i\eta w^-|x) = O(T^2) .$$

Matter fields can be added via the Fock module

$$(d + \Omega_0(v^-, w^+|\mathbf{x})) \star C^i(w^+|\mathbf{x}) \star F = 0$$

Reduction to free CFT_3

The unfolded equation

$$D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^-, w^+|\mathbf{x}, 0) = \mathcal{H}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}(w^+, 0 \mid \mathbf{x}, 0)$$

remains free if

$$\mathcal{T}^{jj} = 0 \qquad \longrightarrow J^{asym} = 0 \qquad \text{or} \qquad J^{sym} = 0$$

depending on whether A-model or B-model is considered. For these cases the model remains free in accordance with the Klebanov-Polyakov Sezgin-Sundell conjecture.

Free models are equivalent to the reductions of the HS theory with respect to P-involution $y \leftrightarrow \bar{y}$ which is possible for the A and B models.

For HS theory with general phase η parameter such reduction is not possible: no realization as a free conformal theory.

Non-Abelian contribution of conformal HS connections has to be taken into account.

Higher-spin theory and quantum mechanics

Rank-one equation can be rewritten in the form

$$\left(ih\frac{\partial}{\partial X^{AB}}+\frac{h^2}{2m}\frac{\partial^2}{\partial Y^A\partial Y^B}\right)\Psi(Y|X)=0\,,\qquad A,B=1,\dots M$$

Algebra of symmetries: algebra of polynomials of $P_A=\frac{\partial}{\partial Y^A}$ and Y^B: conformal HS algebra. $sp(2M)$:

$$K^{AB}=Y^AY^B\,,\qquad L^A{}_B=\{Y^A\,,P_B\}\,,\qquad P_{AB}=P_AP_B$$

Time-like directions in $\mathcal{M}_M$ are associated with positive-definite X^{AB}

$$X^{AB}=tM\delta^{AB}$$

Restriction to t gives M-dimensional Schrodinger equation

$$\left(ih\frac{\partial}{\partial t}+\frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}\right)\Psi(Y|t)=0$$

Y^A are now interpreted as Galilean coordinates.

In unfolded dynamics it is easy to introduce coordinates in which any symmetry h of a given system acts geometrically by introducing a non-zero flat connection of h. Different symmetries require different spaces and connections. Description of the same system in different space-times gives holographically dual theories.

Being obvious in unfolded dynamics, where it refers to the same twistor space (Y^A) in other approaches holographic duality may look obscure.

Maximal finite dimensional symmetry algebra $sph(M|\mathbb{R})$ Valenzuela (2009)

$$T_{AB} = -\frac{i}{2} Y_A Y_B \,, \qquad t_A = Y_A$$

$$[T_{AB}\,, T_{CD}] = C_{BC} T_{AD} + C_{AC} T_{BD} + C_{BD} T_{AC} + C_{AD} T_{BC}$$

$$[T_{AB}\,, t_C] = C_{BC} t_A + C_{AC} t_B \,, \qquad [t_A\,, t_B] = 2i C_{AB}$$

Relativistic and nonrelativistic symmetries of Schrodinger equation belong to $sph(M|\mathbb{R})$. Each symmetry acts geometrically in respective space.

Any HS geometry is holographically dual to some quantum mechanics.

For example, AdS geometry is dual to harmonic potential

$$U(Y) = \frac{1}{2} m\omega^2 Y^A Y^B \delta_{AB}$$

where $-\Lambda \sim \lambda^2$

$$\frac{1}{2} m\omega^2 = \lambda^2 \,.$$

dS geometry is holographically dual to the inverted harmonic potential

not too surprisingly in the context of inflation.

Conclusions

Holographic duality relates theories that have equivalent unfolded formulation: equivalent twistor space description.

AdS_4 HS theory is dual to nonlinear $3d$ conformal HS theory of $3d$ currents

Both of holographically dual theories are HS theories of gravity

Beyond $1/N$

Free boundary theories are dual to truncations of HS theories

Holography of relativistic and nonrelativistic theories

To do

Nonlinear $3d$ conformal HS theory

Actions

Generating functional for correlators

Multiparticle States

AdS_3/CFT_2 **and Gaberdiel-Gopakumar conjecture**

GGI Program

“Higher Spins, Strings and Dualities”

,

Florence, March 18 - May 10, 2013

Organizers:

D.Francia, M.Gaberdiel, I.Klebanov, A.Sagnotti, D.Sorokin, M.Vasiliev

Conformal frame

D **in the twisted adjoint representation is realized by the second-order operator**

$$\{D\,,C\}_* = \left(y^{+\alpha}y^-_\alpha - \frac{1}{4}\frac{\partial^2}{\partial y^{+\alpha}\partial y^-_\alpha}\right)C$$

Fields C inherited from AdS_4 theory are not manifestly conformal.

Conformal frame: Wick star product

$$(f_N \star g_N)(y^\pm) = \int \mu(u^\pm)\,\mathrm{exp}(-u^-_\alpha u^{+\alpha}) f_N(y^+, y^- + u^-) g_N(y^+ + u^+, y^-)$$

$$f_N(y^\pm) = \mathrm{exp}\, -\frac{1}{2}\epsilon^{\alpha\beta}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{+\beta}} f(y^\pm)$$

$$\{D_N\,,\ldots\}_\star = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}}\right) + y^-_\alpha y^{+\alpha} + 1$$

$$T(y^\pm|x) = \mathrm{exp}\, -y^-_\alpha y^{+\alpha} C_N(y^\pm|x)$$

$$\star \quad D_N(T(y^\pm)) = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}} + 2\right)T(y^\pm)$$

Doubling of AdS

$z = 0$ is smooth point in rescaled variables

Continuation $z \to -z$: AdS doubling

Parity automorphism

$$P(z) = -z$$

P-even solution: Neumann boundary condition

P-odd solution: Dirichlet boundary condition

Unfolding as twistor transform

Twistor transform

$$\begin{tikzcd}[column sep=small] & \mathbf{C}(Y|x) \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ \mathbf{M}(x) & & \mathbf{T}(Y)\,. \end{tikzcd}$$

$W^{\Omega}(Y|x)$ are functions on the “correspondence space” C.

Space-time $\mathbf{M}$: coordinates x. Twistor space $\mathbf{T}$: coordinates Y.

Unfolded equations describe the Penrose transform by mapping functions on $\mathbf{T}$ to solutions of field equations in $\mathbf{M}$.

Being simple in terms of unfolded dynamics and the corresponding twistor space $\mathbf{T}$, holographic duality in terms of usual space-time may be complicated requiring solution of at least one of the two unfolded systems: a nontrivial nonlinear integral map.

Global symmetries

Global symmetry transformations that leave a vacuum connection w_0 invariant are described by $\epsilon_{gl}(x)$

$$D_0 \epsilon_{gl} = 0$$

$dim\, h$ independent solutions.

h-module V can be treated as $l^{max}(V)$-module where $l^{max}(V) = End\, V$. Hence $l^{max}(V)$ is the maximal symmetry of the linear unfolded equations with dynamical fields valued in V.

Let W_0^Ω be some solution of the unfolded system may be containing some nonzero p_Ω-forms with $p_\Omega \neq 1$. symmetry parameters $\epsilon_{gl}^\Omega(x)$ satisfy

$$d\varepsilon_{gl}^\Omega + \varepsilon_{gl}^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda}\Big|_{W=W_0} = 0 \,.$$

The 0-form part imposes constraints: global symmetries should leave invariant vacuum values of 0-forms in the system.

Idea of Nonlinear Construction

straightforward construction of nonlinear deformation quickly gets complicated.

trick: doubling of spinors and Klein operators

$$\omega(Y|x) \longrightarrow W(Z;Y;K|x)\,, \qquad C(Y|x) \longrightarrow B(Z;Y;K|x)$$

to be accompanied by equations that determine the dependence on Z_A in terms of "initial data"

$\omega(Y;K|x) = W(0;Y;K|x) = \sum_{ij=1,2} k^i \bar{k}^j \omega^{ij}(Y|x)$

$C(Y;K|x) = B(0;Y;K|x) = \sum_{ij=1,2} k^i \bar{k}^j C^{ij}(Y|x)\,.$

$S(Z,Y,K|x) = dZ^A S_A$ is an connection along Z^A

Klein operators $K = (k, \bar{k})$ generate chirality automorphisms

$$kf(A) = f(\tilde{A})k\,, \quad \bar{k}f(A) = f(-\tilde{A})\bar{k}\,, \qquad A = (a_\alpha\,, \bar{a}_{\dot\alpha}) : \quad \tilde{A} = A = (-a_\alpha\,, \bar{a}_{\dot\alpha})$$

$$P(Y) = P^{\alpha\dot\alpha} y_\alpha \bar{y}_{\dot\alpha} \quad \longrightarrow \quad \tilde{P}(Y) = -P(Y)\,, \qquad \tilde{M}(Y) = M(Y)\,.$$

Nonlinear HS Equations

HS start-product

$$(f \star g)(Z,Y) = \int dSdT f(Z+S, Y+S) g(Z-T, Y+T) \exp -iS_\nu T^\nu$$

$$[Y_A, Y_B]_\star = -[Z_A, Z_B]_\star = 2iC_{AB}\,, \qquad Z-Y\ :\ Z+Y \text{ **normal ordering**}$$

Inner Klein operators:

$$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar\kappa = \exp iz_{\dot\alpha} y^{\dot\alpha}\,, \qquad \kappa \star f = \tilde{f} \star \kappa\,, \qquad \kappa \star \kappa = 1$$

HS equations:

$$\mathcal{W} = (d+W) + S\,, \qquad W = dx^n W_n\,, \qquad S = dz^\alpha S_\alpha + d\bar z^{\dot\alpha} \bar S_{\dot\alpha}$$

$$\mathcal{W} \star \mathcal{W} = i(dZ^A dZ_A + dz^\alpha dz_\alpha F(B) \star k \star \kappa + d\bar z^{\dot\alpha} d\bar z_{\dot\alpha} \bar F(B) \star \bar k \star \bar\kappa)\,,$$

$$\mathcal{W} \star B = B \star \mathcal{W}$$

Manifest gauge invariance

$$\delta\mathcal{W} = [\varepsilon, \mathcal{W}]_\star\,, \qquad \delta B = \varepsilon \star B - B \star \varepsilon\,, \qquad \varepsilon = \varepsilon(Z;Y;K|x)$$

Integrating out Space-Time

$x - Z$ **decomposition:**

$$\begin{cases} dW + W \star W = 0 \\ dB + W \star B - B \star W = 0 \\ dS + W \star S + S \star W = 0 \\ S \star B - B \star S = 0 \\ S \star S = i(dZ^A dZ_A + dz^\alpha dz_\alpha F(B) \star k \star \kappa + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \bar{F}(B) \star \bar{k} \star \bar{\kappa}) \end{cases}$$

Nontrivial equations are free of space-time differential d: **Space-time dependence is locally pure gauge:**

$$W(Y, Z|x) = g^{-1}(Y, Z|x) * dg(Y, Z|x)$$

$$B(Y, Z|x) = g^{-1}(Y, Z|x) * B_0(Y, Z) * g(Y, Z|x)$$

$$S(Y, Z|x) = g^{-1}(Y, Z|x) * S_0(Y, Z) * g(Y, Z|x)$$

HS equations describe two dimensional fuzzy hyperboloid in noncommutative space of Y_A **and** Z_A. **Its radius depends on HS curvature** $B(x)$.

$d = 3$: **no dotted spinors**

Holographic conformal currents

Equation on $3d$ 0-forms

$$D^{tw}_{\mathbf{x}} T(y,\bar{y}|x) = d_{\mathbf{x}} T(y,\bar{y}|x) + 4 d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} T(y,\bar{y}|x) = 0$$

describes two sets of conserved currents of al spins $s \geq 0$ distinguished by their symmetry

$$J^{sym}(y,\bar{y}|x) = T(y,\bar{y}|x) + T(y,\bar{y}|x)\,, \qquad J^{asym}(y,\bar{y}|x) = T(y,\bar{y}|x) - T(y,\bar{y}|x)\,,$$

$$\Delta(J^{sym}(0,0|x)) = 1\,, \qquad \Delta(J^{asym}(0,0|x)) = 2\,.$$

Invariant functionals via Q–cohomology

Equivalent form of compatibility condition

$$Q^2 = 0\,, \qquad Q = G^{\Omega}(W)\frac{\partial}{\partial W^{\Omega}}$$

Q-manifolds

Hamiltonian-like form of the unfolded equations

$$dF(W(x)) = Q(F(W(x))\,, \qquad \forall F(W)\,.$$

Invariant functionals

$$S = \int L(W(x))\,, \qquad QL = 0 \qquad \text{(2005)}$$

$L = QM$: total derivatives

Actions and conserved charges: Q cohomology

for off-shell and on-shell unfolded systems, respectively

$2M$–form

$$\Omega^{2M}(T) = \Big(d\mathcal{W}_A \wedge \Big(i\,\mathcal{W}_B d\,X^{AB} - d\,Y^A\Big)\Big)^M \;\widetilde{T}(\mathcal{W},\,Y\,|X)$$

is closed in $\mathcal{M}_M \times \mathbb{R}^M(\mathcal{W}_B) \times \mathbb{C}^M(Y^A)$

The charge

$$q = q(T) = \int_{\Sigma^{2M}} \Omega^{2M}(T)$$

is independent of local variations of a $2M$-dimensional surface Σ^{2M}.

Remarkable output: conserved charges can be expressed as integrals over the twistor space $\mathbb{T}$

Solutions of current equation form a commutative algebra

$$\eta(\mathcal{W},\,Y\,|X) = \varepsilon(\mathcal{W}_A,\,Y^C - i\,X^{CB}\,\mathcal{W}_B)\,, \qquad \widetilde{T}_\eta(\mathcal{W},\,Y\,|X) = \eta(\mathcal{W},\,Y\,|X)\widetilde{T}(\mathcal{W},\,Y\,|X$$

$\eta(\mathcal{W},\,Y\,|X)$ is a polynomial parameter representing global HS symmetry. $q(\widetilde{T}_\eta)$ with various $\eta(\mathcal{W},\,Y\,|X)$ generate complete set of conformal HS conserved charges. $M=2$: all conserved charges built from bilinears of free $3d$ massless fields.

Higher rank as higher dimension

A rank–r field in $\mathcal{M}_M \sim$ a rank–one field in $\mathcal{M}_{rM}$ with coordinates X^{AB}_{ij}.

$$Y_i^A \to Y^{\widetilde{A}}, \qquad \widetilde{A} = 1 \dots rM$$

Embedding of $\mathcal{M}_M$ into $\mathcal{M}_{rM}$

$$X^{AB}_{11} = X^{AB}_{22} = \dots = X^{AB}_{rr} = X^{AB}$$

$3d$ **conformal currents:**

a rank-two field in $\mathcal{M}_2$ $(d = 3) \sim$ rank-one field in $\mathcal{M}_4$ $(d = 4)$.

A single rank-one field in $\mathcal{M}_4$ **describes all** $4d$ **conformal fields.**

Realization of Flato-Fronsdal Thm

What if the system is deformed by a potential? Formally, this does not affect the consideration much. In presence of potential $U(Y)$ the equation

$$\left(ih\frac{\partial}{\partial t}+\frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}-U(Y)\right)\Psi(Y|t)=0$$

remains linear, hence exhibiting infinite symmetries. It can be interpreted as flatness condition

$$D\Psi(Y|t)=0\,,\qquad D=dt\frac{\partial}{\partial t}+\Omega\,,\qquad \Omega=ih^{-1}dtH\,,\qquad H=-\frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}$$

In the $1d$ case with the single coordinate t, any connection is flat. Hence it can be represented in the pure gauge form which is simply

$$\Omega=\exp -ih^{-1}Ht\,d\,\exp ih^{-1}Ht$$